Riding from Montréal to Los Angeles

Aaron Montgomery
montgoaa@cwu.edu

Central Washington University

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Riding from ~~Montréal~~ *Seattle* to ~~Los Angeles~~ *New York*

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Mathematics of Tabletop Games

Where does this material come from, and who is the intended audience?

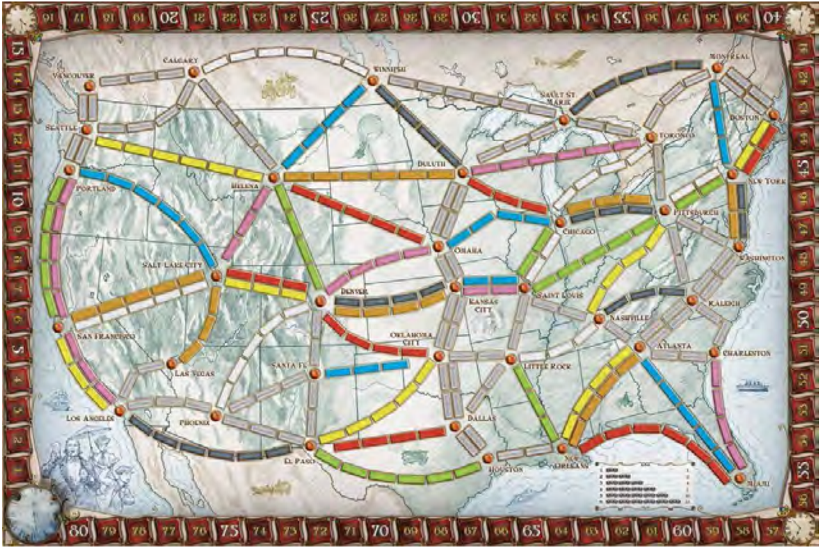
- Two courses in tabletop games:
  - A 1-credit lower-division Enrichment Seminar (2019);
  - A 5-credit upper-division Topics Course (2021).
- Material for those classes was collected into a book (July 2024).
- The content of the courses and the book are intended to provide starting points for undergraduate exploration of discrete math topics.

Outline

- Introduction to *Ticket To Ride*.
- What the pfaff is a generating functions?
- What the pfaff is an adjacency matrices?
- What happens if they have a child?
- "This is all a bit hand-wavy" — Emilie Purvine

Ticket To Ride: Board

Ticket To Ride is a tabletop game played on this board.



Ticket To Ride: Rules

You earn points in *Ticket To Ride* by:

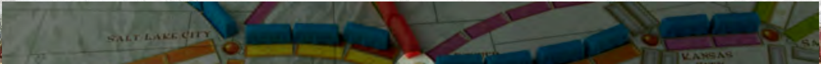
- Playing sets of cards with the same color to claim routes:

Cards	Points Earned
1	1
2	2
3	4
4	7
5	10
6	15

- Linking pairs of cities on ticket cards.

Ticket To Ride: Scoring Example

- Here, Red, the red player, is taking a turn.
- Red has the "Denver - El Paso" Ticket.
- They can discard three Green cards
- to claim the route from Phoenix to El Paso,
- earning 4 points for a three-car route.
- In addition to the 4 points for completing the ticket.



A *Ticket To Ride* Question

- In *Ticket To Ride*, there are 110 color cards:
- twelve cards of eight colors and fourteen wildcards.
- During a player's turn, five cards are face-up cards from which they can draw.
- How many possible sets of face-up cards are possible?

Generating Functions: Technique

- We can use generating functions to count the number of possible sets of face-up cards.
- The technique uses variables (r, b, t etc.) as catalysts. The variables don't represent a quantity, but help facilitate the calculation.
- Multiplication represents repeated selections, for example:
  - $r = r^1$ represents selecting a red card,
  - r^2 represents selecting two red cards,
  - r^3 three red cards,
  - $1 = r^0$ selecting no red cards.
- Summing represents an exclusive choice. For example, selecting zero to five red cards would be represented as the expression:

$$r^0 + r^1 + r^2 + r^3 + r^4 + r^5.$$

Generating Functions: Technique

- Continuing the example, selecting zero to five red cards and zero to five blue cards would generate the expression:

$$(1 + r + r^2 + r^3 + r^4 + r^5) \times (1 + b + b^2 + b^3 + b^4 + b^5)$$

- Which expands to:

$$1 + b + b^2 + b^3 + b^4 + b^5 + r + rb + rb^2 + \cdots + r^5 b^5$$

- Removing the color information, we obtain

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \mathbf{6x^5} + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}$$

indicating that there are six ways to select five cards.

Generating Functions: *Ticket To Ride* Answer

- The selection of all possible sets from the deck is encoded as

$$(1 + x + \cdots + x^{12})^8 \times (1 + x + \cdots + x^{14})$$

- For lower-division students, a computer algebra system can be asked to do the calculation:

$$1 + \cdots + \mathbf{1287x^5} + \cdots + x^{110}$$

indicating 1287 possible sets of size five.

- For upper-division students, a formal series can be used to determine the coefficient:

$$\binom{13}{5} = 1287.$$

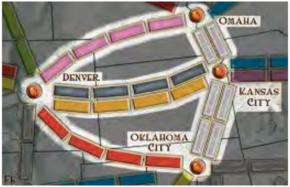
Another *Ticket To Ride* Question

- How can we calculate the shortest path between pairs of cities on the map?
- In particular, what is the shortest path from Seattle to New York?

Adjacency Matrices: Technique

- We can compute the paths between all pairs of cities simultaneously using an adjacency matrix, A.
- The (i, j) entry of A is the number of routes between city i and city j.

Adjacency Matrices: Technique



$$A = \begin{array}{c} \\ \vdots \\ Oma \\ KC \\ Den \\ OkC \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \cdots & Oma & KC & Den & OkC & \cdots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & 2 & 1 & 0 & \cdots \\ \cdots & 2 & 0 & 2 & 2 & \cdots \\ \cdots & 1 & 2 & 0 & 1 & \cdots \\ \cdots & 0 & 2 & 1 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

Adjacency Matrices: *Ticket To Ride* Answer

- Matrix multiplication can be used to track paths, as the entries of A^k are the number of paths using k routes between pairs of cities.
- For $k = 4$, we find no paths between Seattle and New York:

$$A^4 = \begin{array}{c} \\ \vdots \\ NY \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & Sea & \cdots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & 0 & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}$$

- For $k = 5$, we find 14 paths between Seattle and New York:

$$A^5 = \begin{array}{c} \\ \vdots \\ NY \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & Sea & \cdots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \iddots \\ \cdots & 14 & \cdots \\ \iddots & \vdots & \ddots \end{pmatrix} \end{array}$$

More *Ticket To Ride* Questions

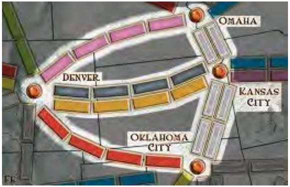
Some unanswered questions about the 14 paths between Seattle and New York:

- How many victory points do the paths score?
- How many cards are required for each path?
- What colors are required for each path?

Generating Adjacency Matrix Functions: Technique

- We can use generating functions to include more information in the adjacency matrix.
- We can use a variable for victory points (v) to track the points earned by each path.
- We can use variables for each color (r for red, b for blue, x for wild, etc.) to track card requirements.
- You can even track the cities using the adjacency matrix with more work by including a factor identifying cities in each matrix entry (see Python files on `github` for details).

Generating Adjacency Matrix Functions: Technique



$$
A = \begin{array}{c} \\ \vdots \\ Oma \\ KC \\ Den \\ OkC \\ \vdots \end{array}
\begin{array}{c} \cdots \quad Oma \quad KC \quad Den \quad OkC \quad \cdots \\
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 0 & 2xv & p^4v^7 & 0 & \cdots \\
\cdots & 2xv & 0 & k^4v^7 + t^4v^7 & 2x^2v^2 & \cdots \\
\cdots & p^4v^7 & k^4v^7 + t^4v^7 & 0 & r^4v^7 & \cdots \\
\cdots & 0 & 2x^2v^2 & r^4v^7 & 0 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

Gen Ad Ma Funcs: *Ticket To Ride* Answers

Now A^5 contains significantly more information.

$$\begin{aligned} A^5[NY, Sea] = \; & v^{53}b^3p^6t^6x^3y^6 + v^{51}b^7k^5x^6y^6 \\ & + v^{51}b^3k^5w^6x^{10} + v^{49}g^2p^6t^6x^2y^6 \\ & + v^{49}p^6t^6w^2x^2y^6 + v^{48}b^3k^5t^6x^3y^6 \\ & + v^{40}g^2k^3r^3t^6y^6 + v^{40}g^2r^3t^9y^6 \\ & + v^{40}k^3r^3t^6w^2y^6 + v^{40}r^3t^9w^2y^6 \\ & + v^{38}b^4g^2k^3r^5y^6 + v^{38}b^4g^2r^5t^3y^6 \\ & + v^{38}b^4k^3r^5w^2y^6 + v^{38}b^4r^5t^3w^2y^6 \end{aligned}$$

Ticket To Ride: Most Victory Points

The highest-scoring path is represented by

$$v^{53} b^3 p^6 t^6 x^3 y^6.$$

Seattle–Helena–Duluth–Toronto–Montréal–New York

This provides $53/24 \approx 2.21$ points per card. If we include the value of the ticket, we get $53 + 22 = 75$ points, $75/24 = 3.125$ points per card.

Ticket To Ride: Most Points Per Card

The paths scoring the most points per card (≈ 3.23) are represented by

$$v^{49}g^2p^6t^6x^2y^6 + v^{49}p^6t^6w^2x^2y^6.$$

Seattle–Helena–Duluth–Toronto–Pittsburgh–New York

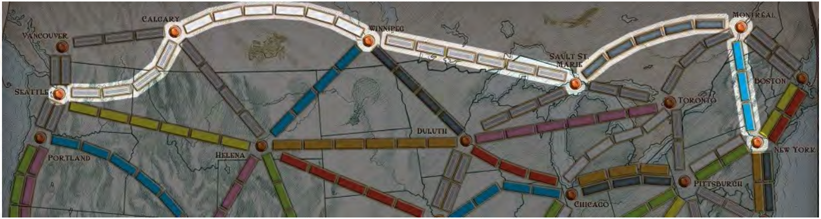
Ticket To Ride: Easiest to Complete

Also attractive is the route represented by

$$v^{51} b^3 k^5 w^6 x^{10}$$

because it uses 10 any color spaces, making it easier to complete.

Seattle–Calgary–Winnipeg–Sault St. Marie–Montréal–New York



Gen Ad Ma Funcs: Limitations

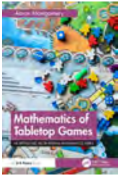
- Adjacency Matrices find walks, not paths, higher powers of A counts things like Seattle–Helena–Seattle–Helena–Duluth–Toronto–Pittsburgh–NY.
- While matrix multiplication and polynomial expansion are not exponentially complex, raising large matrices of polynomials to high powers and extracting coefficients takes time and memory to compute.

Ticket To Ride: Other Considerations

- Time required to collect required sets (see Witter & Lyford).
- Congestion and blocking of routes as other players claim routes (see Witter & Lyford).
- The synergies between tickets.

References

- R. Teal Witter & Alex Lyford: *Applications of Graph Theory and Probability in the Board Game Ticket To Ride*, Foundations of Digital Games, September 15–18, 2020, Bugibba, Malta.
- Aaron Montgomery *Mathematics of Tabletop Games*.



- Combinatorics
- Geometry
- Group Theory
- Graph Theory
- Probability
- Game Theory
- Auctions
- Deduction
- Number Theory

- `http://www.monsterworks.com/ludi/`
- `https://github.com/monsterworks-ludi/tabletop`

Extra Stuff

The following slides were not part of the talk proper, but were supplemental material or based on questions asked at the talk.

Traditional Generating Functions

To compute the coefficient on x^5 in

$$(1+x+\cdots+x^{12})^8(1+x+\cdots+x^{14}),$$

we compute

$$\begin{aligned}(1+x+\cdots+x^{12})^8(1+x+\cdots+x^{14})+O(x^{13})\\ =\left(\sum_{n=0}^{\infty}x^n\right)^9=\frac{1}{(1-x)^9}=\sum_{n=0}^{\infty}\binom{n+8}{n}x^n.\end{aligned}$$

Extracting the coefficient on x^5 arrives at $\binom{13}{5}=1287$.

Optimal Routes

Doing a depth-first search of all paths requiring 45 or fewer pieces for each ticket, we find the following:

Points/Card	Ticket	Points	Cards	Hops	WL
3.35	Miami, Los Angeles	67	20	4	3
3.23	New York, Seattle	71	22	5	2
3.20	Montreal, Vancouver	64	20	4	1
2.92	Sante Fe, Vancouver	38	13	4	14
22 rows omitted					
2.40	Atlanta, Montreal	96	40	9	25
2.38	El Paso, Denver	33	14	3	18
2.36	New Orleans, Chicago	106	45	10	26
2.29	Atlanta, New York	16	7	3	28

WL column refers to Witte & Lyford's ranking of the routes in a 2-player game based on simulated play.

Maximum Face-up Wildcards

The actual rules disallow having more than two wildcards in the face-up cards. This can also be done by generating functions, expanding

$$(1 + x + \cdots + x^{12})^8 \times (1 + x + x^2).$$

To do this without a computer, one would use the traditional generating function technique on the first factor and then do three manual multiplications.
The final answer will be 1178 instead of 1287.